

**NEXT-TO-LEADING ORDER EVOLUTION OF  
STRUCTURE FUNCTIONS AT SMALL  $x$  AND LARGE  $Q^2$  \***

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**Abstract**

We show that a unified approach to the perturbative evolution of structure functions which sums all logarithms of  $Q^2$  and  $1/x$  at leading and next-to-leading order yields results in full agreement with the 1993 HERA data for  $F_2$ . This makes it possible to determine  $\alpha_s$  surprisingly accurately from these data alone.

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At small  $x$  the structure function  $F_2^p(x, Q^2)$  measured recently at HERA [1][2], depends in effect on two scales,  $Q^2$  and  $s = (\frac{1-x}{x})Q^2$ , instead of just the single scale  $Q^2$  relevant at large  $x$ . The renormalization group equations which govern the perturbative evolution of singlet parton distribution functions thus describe evolution in both  $s$  and  $Q^2$  (or equivalently  $1/x \simeq s/Q^2$  and  $t \equiv \ln Q^2/\Lambda^2$ ), and should sum up all leading (and subleading) logarithms of both  $Q^2$  and  $s$  (or  $Q^2$  and  $1/x$ ). In fact it is possible to construct unified perturbative evolution equations for leading twist singlet parton distribution functions which are straightforward generalizations of the usual singlet Altarelli-Parisi equations, but with new splitting functions which explicitly incorporate all the logarithms of  $1/x$ . The resulting equations[3] may be used to calculate perturbative evolution down to arbitrarily small values of  $x$ , provided only that  $Q^2$  is sufficiently large that higher twist effects may be ignored. All nonperturbative effects (at both large and small  $x$ ) are then factorized into the initial condition at  $t_0$ .

The singlet splitting functions in the the double leading expansion scheme, which treats the two scales symmetrically (see ref.[3]), take the form  $P^{ij}(x, t) = P_{\text{LO}}^{ij}(x) + \frac{\alpha_s(t)}{2\pi} P_{\text{NLO}}^{ij}(x, t) + \dots$ , where  $P_{\text{LO}}^{ij}(x) = P_1^{ij}(x) + P_s^{ij}(x, t)$ ,  $P_{\text{NLO}}^{ij} = P_2^{ij}(x) + P_{ss}^{ij}(x, t)$ ,  $P_1^{ij}(x)$  and  $P_2^{ij}(x)$  are the usual one and two loop splitting functions, while  $P_s^{ij}(x, t)$  and  $P_{ss}^{ij}(x, t)$  are (convergent) sums of leading and subleading singularities respectively: for  $x > x_0$  they are all zero, while for  $x < x_0$   $P_s^{gg}(x, t) = P_{ss}^{gg}(x, t) = 0$ , but

$$\begin{aligned} xP_s^{gg}(x, t) &= \frac{C_A}{C_F} xP_s^{gq}(x, t) = 2C_A \sum_{n=4}^{\infty} a_n \frac{1}{(n-1)!} \lambda_s(t)^{n-1} \xi^{n-1}, \\ xP_{ss}^{gq}(x, t) &= \frac{C_A}{C_F} xP_{ss}^{qq}(x, t) = \frac{2}{3} T_R n_f \sum_{n=3}^{\infty} \tilde{a}_n \frac{1}{(n-2)!} \lambda_s(t)^{n-1} \xi^{n-2}, \\ xP_{ss}^{gg}(x, t) &= \frac{C_A}{C_F} xP_{ss}^{gq}(x, t) = 2C_A \sum_{n=3}^{\infty} b_n \frac{1}{(n-2)!} \lambda_s(t)^{n-1} \xi^{n-2}, \end{aligned} \quad (1)$$

where  $\lambda_s(t) \equiv 4 \ln 2 \frac{C_A}{\pi} \alpha_s(t)$ ,  $\alpha_s(t)$  is the two-loop running coupling, and  $\xi = \log \frac{x_0}{x}$ .

The coefficients  $(a_n, \tilde{a}_n)$  may be found in ref.[3]: they are computed using expressions derived in ref.[4]. The coefficients  $b_n$  (and the color-charge relation between  $P_{ss}^{gg}$  and  $P_{ss}^{gq}$ ) can be fixed uniquely at NLO by requiring momentum conservation:  $b_n = -a_n - \frac{T_R n_f}{3C_A} \tilde{a}_n$ . This fixes the arbitrariness implicit in alternative schemes in which momentum is conserved separately for quarks and gluons[5], and furthermore makes the explicit computation of  $P_{ss}^{gg}$  and  $P_{ss}^{gq}$  unnecessary at NLO: subleading corrections to the BFKL kernel would only be necessary for NNLO computations. The parameter  $x_0$ , which marks the boundary

between large and small  $x$  regions, must eventually be fixed empirically: we vary it in the range  $0 \leq x_0 \leq 0.1$ , considering larger values of  $x_0$  to be unreasonable.

Each of the series for the splitting functions is uniformly convergent on any closed interval of  $x$  which excludes the origin, and can thus be used to extend perturbative evolution down to arbitrarily small values of  $x$ [3]. It follows that their Mellin transforms (the anomalous dimensions  $\gamma_N$ ) have no physically relevant singularities beyond those present at fixed order in  $\alpha_s$  (so in particular any cuts generated by all order resummations may in practice be ignored).

The asymptotic behaviour of the parton distribution functions, as solutions to these unified evolution equations, depends on the particular limit adopted in the  $x$ - $t$  plane. In the Bjorken limit  $t \rightarrow \infty$ , with  $x$  fixed, the most important parts of the splitting functions are  $P_1$  to LO and  $P_2$  at NLO. The dependence on  $x$  is essentially nonperturbative, being given by the initial condition at  $t_0$ , while the dependence on  $t$  is perturbative. In the Regge limit  $\xi \rightarrow \infty$  at fixed  $t$ , the most important parts of the splitting functions are the leading and subleading singularities  $P_s$  and  $P_{ss}$ . In this limit  $F_2$  eventually rises as  $x^{-\lambda(t)}$ , the power  $\lambda(t)$  being given by Regge theory for  $t \sim t_0$ , but calculable perturbatively for  $t$  sufficiently large ( $\lambda(t) = \lambda_s(t)$ )[3]. Finally, in the double scaling limit  $\xi \rightarrow \infty$ ,  $\zeta \equiv \ln \frac{t}{t_0} \rightarrow \infty$ ,  $\xi/\zeta$  rising as some power of  $\zeta$ , but slower than an exponential, the asymptotic form of singlet parton distribution functions may be determined completely in perturbation theory[6][7][3]. In particular the structure function  $F_2(x, t)$  then grows as  $R_F^{-1} \equiv \mathcal{N} \sigma^{-\frac{1}{2}} \rho^{-1} e^{2\gamma\sigma - \delta\zeta}$ , where  $\sigma \equiv \sqrt{\xi\zeta}$ ,  $\rho \equiv \sqrt{\xi/\zeta}$ ,  $\gamma = \frac{6}{5}$  and  $\delta = \frac{61}{45}$ . This leads to a double asymptotic scaling in  $\sigma$  and  $\rho$  which is indeed seen experimentally [7][8](see fig. 1). In the double scaling limit (and thus in the kinematic regime studied at HERA) the most important parts of the splitting functions are thus the pivotal leading singularity of  $P_1$  (which gives  $\gamma$ ) and the subleading term (which gives  $\delta$ ) at LO, and the leading singularity of  $P_2$  at NLO.

Two loop calculations of  $F_2^p$  in the double scaling region were performed in ref.[9]: here we give similar calculations in the double leading scheme. The solution of the NLO evolution equations is straightforward, although care must be taken to linearize all subleading corrections, to avoid spurious sub-subleading terms in the solution. As in [9] we use as input the MRS distribution D0' with a small- $x$  tail parameterized by the single parameter  $\lambda(Q_0)$  at the starting scale  $Q_0$ . The two parameters  $\lambda(Q_0)$  and  $\alpha_s$  are then fitted by minimising the total  $\chi^2$  of the evolved distribution to the HERA data. The resulting best-fit parameters are given in table 1, and the corresponding distributions displayed in figure 1, in the form of two scaling plots[7]. The two loop curve (i.e. that with  $x_0 = 0$ )

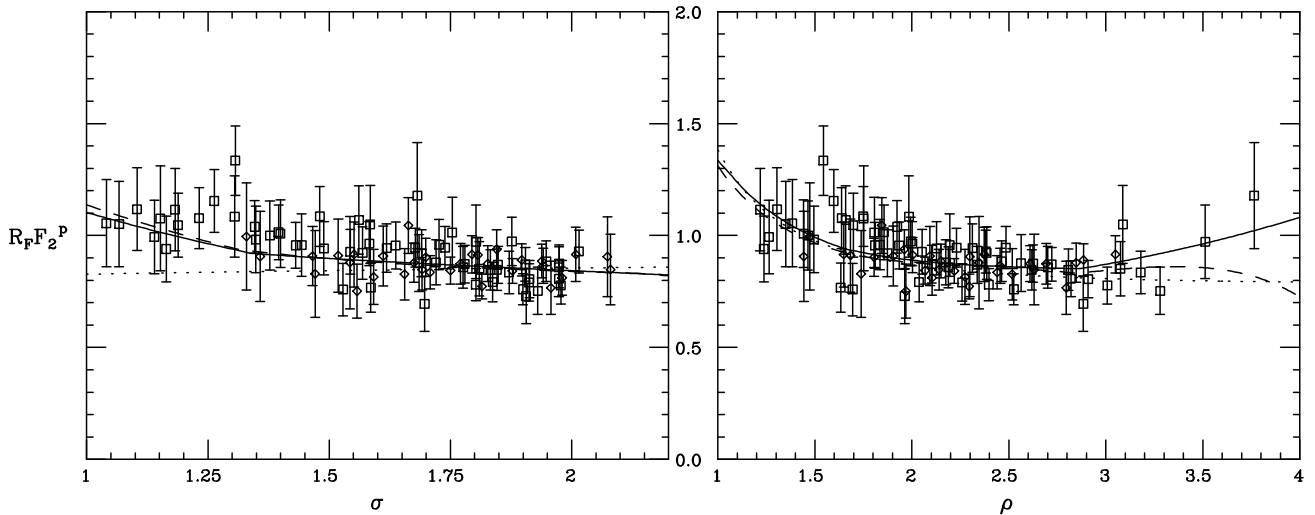


Figure 1: Scaling plots of  $R_F F_2^p$  vs  $\sigma$  and  $\rho$ . All 1993 HERA data passing the cuts  $\sigma, \rho > 1$ ,  $Q^2 < (2m_b)^2$  are included, and  $R_F$  is thus evaluated with  $n_f = 4$ . The diamonds and squares are ZEUS[1] and H1[2] data, renormalized by 1.02 and 0.95 respectively. The curves are the simple double scaling prediction [7] (dotted), and two NLO double leading calculations with extremal values of  $x_0$ :  $x_0 = 0$  (i.e. two loops only) (solid) and  $x_0 = 0.1$  (dashed). The curves in the  $\sigma$ -plot have  $\rho = 2.2$ : those on the  $\rho$ -plot  $\sigma = 1.7$ .

now fits the data significantly better than the scaling curve (which has  $\chi^2 = 71/121$ ), essentially because the most singular part of the two loop correction reduces the slope of the  $\sigma$  plot a little at lower values of  $\sigma$ , a tendency which is evident in the data. The curve with  $x_0 = 0.1$  is almost identical to that with  $x_0 = 0$ , differing only at large values of  $\rho$ ; as explained in ref.[3], most of the effect of the higher order logarithmic singularities can be absorbed by suitably adjusting the boundary condition, either by changing  $Q_0$  at fixed  $\lambda$ , or (as here)  $\lambda$  at fixed  $Q_0$ . If one were to keep the boundary condition fixed one might instead have the impression that the higher loop singularities have a large effect on  $F_2$  in the HERA region.

Whereas the higher order singularities only affect  $F_2$  close to the nonperturbative boundary at  $t = t_0$ , they can have a relatively large effect on the size of the quark and gluon distributions at small  $x$  and large  $t$ . To search for this effect (and, if it is found, to measure  $x_0$ ) it would be very interesting to have an independent measurement of the gluon distribution, either through  $F_L$  or heavy quark production. If  $x_0$  is as large as 0.1, such a

$x_0$	norms		$\lambda(2 \text{ GeV})$	$\alpha_s(M_Z)$	$\chi^2$
0	102%	95%	$-0.23 \pm 0.03$	$0.122 \pm 0.002$	58.4/120
0.1	99%	93%	$-0.06 \pm 0.06$	$0.114 \pm 0.004$	64.9/120

Table: Fitted parameters and  $\chi^2$  for the two fitted curves in figure 1. Statistical and systematic errors for each data have been added in quadrature, and the normalizations of the two experiments fitted within their stated uncertainties of  $\pm 3.5\%$  and  $\pm 4.5\%$  respectively[1][2].

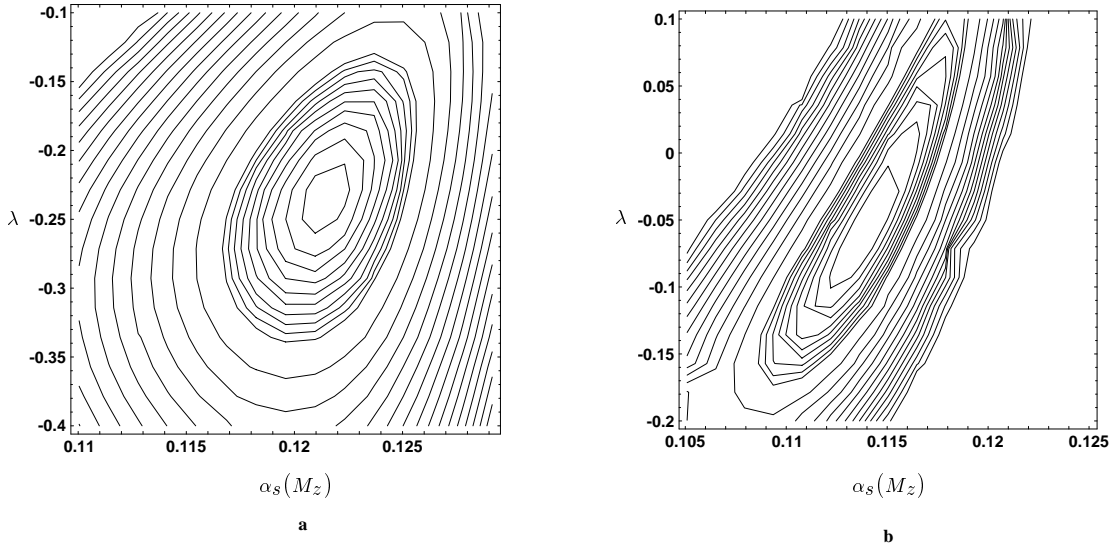


Figure 2: Contour plots of the  $\chi^2$  in the  $\lambda$ - $\alpha_s$  plane, for the two fits shown in the table: a)  $x_0 = 0$ ; b)  $x_0 = 0.1$ . Contours after the first ten are at intervals of five units.

measurement should find a significantly smaller gluon distribution than that expected at two loops.

Since the rise in  $F_2$  at small  $x$  and large  $Q^2$  is being driven essentially by the triple gluon vertex, it depends rather strongly on the value of  $\alpha_s$ . It should thus be a good place to measure  $\alpha_s$ , requiring far lower statistics than at large  $x$ , where one has to search instead for small violations of Bjorken scaling[10]. Indeed the 1993 HERA data are already sufficient for such a determination, as is apparent from the two parameter fits of shown in the table (see also fig. 2), and the fact that the two loop correction can be seen in the data. A preliminary error analysis gives

$$\alpha_s(M_Z) = 0.120 \pm 0.005(\text{exp}) \pm 0.010(\text{th}) : \quad (2)$$

a more detailed determination will be presented elsewhere. As at large  $x$  the theoretical error is dominated by the renormalization scale uncertainty, with an additional uncertainty here due to the unknown value of  $x_0$ . Higher twist effects however seem to be very small.

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